2021年1月6日 王冬

Prop 9.2.14. Assume that $|I|\ge 2$, and that $\mathcal{G}$ satisfies (CG3′)

Let $X\in\mathcal{X}$, $i,j\in I$, with $i\ne j$.

and $\mathcal{Y}=\{r_{i_1}\cdots r_{i_k}(X) \mid k\ge 0,\ i_1,\dots,i_k\in\{i,j\}\}$ $\qquad Y=r_{i_1}\cdots r_{i_k}(X)$

Then $\bar{m}^Y_{ij}=\bar{m}^Y_{ji}=\bar{m}^X_{ij}$ for any $Y\in\mathcal{Y}$ constant $\qquad X=r_i$

$i, j, i, j, i, j \cdots\cdots$

example 9.1.2 $I=\{1,2\}$, $\mathcal{X}=\{X_1,X_2\}$, $r_1(X_1)=X_2$

$r_1(X_2)=X_1$, $r_2=id$

$A^{X_1}=\begin{pmatrix}2&-1\\-3&2\end{pmatrix}$, $A^{X_2}=\begin{pmatrix}2&-1\\-4&2\end{pmatrix}$ $\qquad 1\ 2\ 1$

$K=(1,2,1,2,\cdots\cdots)$ $\qquad s_1^{X_1}s_2^{X_2}(X_1)$

$s_1^{X_1}s_2^{X_2}(\alpha) \Leftarrow \alpha=a_n\alpha_1+b_n\alpha_2$ $\qquad s_2(X_1)$

$a_{n+1}=2a_n-b_n$

$b_{n+1}=3a_n-b_n$ $\qquad s_2 s_1$

β长度

$\beta^{X_1K}_k$	a_n	b_n
1	1	0
3	2	3
5	1	3
~~7~~	~~-1~~	~~0~~
~~9~~	~~-2~~	~~-3~~
~~11~~	~~1~~	~~3~~
~~13~~	~~1~~	~~0~~

$\beta^{X_2K}_k$	a_n	b_n
2	1	1
4	1	2
6	0	1
~~8~~	~~-1~~	~~-1~~
~~10~~	~~-1~~	~~-2~~
~~12~~	~~0~~	~~-1~~
~~14~~	~~1~~	~~1~~

∞

$\bar{m}^{X_1}_{12}=6$

$s_2 s_1(a_n\alpha_1+b_n\alpha_2)=(b_n-a_n)\alpha_1+(3b_n-4a_n)$

β	a_n	b_n
1	0	1
3	1	3
5	2	5
	3	7
	4	9

$\bar{m}^{X_1}_{21}=\infty$?

Ex 9.1.28, $\bar{m}^Y_{23}=3$ $\quad \bar{m}^X_{32}=3$, $r_2(X)=X$ $\quad r_3(X)=X$

Pf: ① If $\bar{m}^Y_{ij}=\bar{m}^Y_{ji}=\infty$, for any $Y\in\mathcal{Y}$ (if $k=0$, include X)

then we are done

和? $\bar{m}^X_{ij}$

② Assume that $\bar{m}^X_{ij}<\infty$ and that $\bar{m}^Y_{ij}, \bar{m}^Y_{ji}\ge\bar{m}^X_{ij}$ for $\forall Y\in\mathcal{Y}$

不妨设 $\qquad (\bar{m}^X_{ij}, \bar{m}^X_{ji})$, $\bar{m}^{r_i(X)}_{ij}$

轨道里的最小值 取X

We prove that $\bar{m}^Y_{i'j'}=\bar{m}^X_{ij}$ for $(Y,i',j')=\tau(X,i,j)$

and for $(Y,i',j')=\sigma(X,i,j)$

where τ and σ are as in (9.2.1)

③ Let $m=\bar{m}^X_{ij}$, $Y=r_i(X)$, and $K=(j,i,j,i\cdots)=(j_1,\cdots,j_m)$

Then K is Y-reduced since $\bar{m}^Y_{ji}\ge\bar{m}^X_{ij}$

In particular, $\beta^{Y,K}_{m-1}\in\mathbb{N}_0^I$ (by (CG3′))

$\beta^{Y,K}_m=\alpha_i$ by Lem 9.2.13

Then $\beta^{Y,K}_{m+1}\notin\mathbb{N}_0^I$ by Lem 9.2.11, ($m+1$ not reduced)

$\therefore\ \bar{m}^Y_{ji}=m$ (by (CG3′))

④ Let now $Z=r_{j_m}\cdots r_{j_1}(Y)$ and $K'=(j_m,\cdots j_1, i)$

Then K' isn't Z-reduced by lemma 9.2.5 (倒过来可以走)

since $(i,j_1,\cdots j_m)$ isn't X-reduced

because of $m=\bar{m}^X_{ij}$

On the other hand, $\bar{m}^Z_{j_m j_{m-1}}\ge\bar{m}^X_{ij}=m$,

Thus $\bar{m}^Z_{j_m j_{m-1}}=m$

$\tau^{m+1}(Z,j_m,j_{m-1})=\tau(Y,i,j)=(X,j,i)=\sigma(X,i,j)$

$\bar{m}^X_{ji}=m$, proves the Prop $\quad\square$

Lem 9.2.15 Assume that $|I|\ge 2$, let $X\in\mathcal{X}$, $i,j\in I$

with $i\ne j$, $K=K^X_{ij}$, and $m=\bar{m}^X_{ij}$, $m<\infty$

① If $\mathcal{G}$ satisfies (CG3′), then $\beta^{X,K}_1=\alpha_i$,

$\beta^{X,K}_m=\alpha_j$, and $id_X(s_is_j)^m(\alpha_i)=\alpha_i$,

$id_X(s_is_j)^m(\alpha_j)=\alpha_j$ $\qquad id_X$ 的原因：终点固定 P318

s_i 上标

$s_i^{(X)} s_j$

② If $\mathcal{G}$ satisfies (CG3′) and (CG4′)

then $id_X(s_is_j^m)=id_X$

Pf: (1) ① Assume that $\mathcal{G}$ satisfies (CG3′)

Then $\bar{m}^{r_i(X)}_{ji}=\bar{m}^X_{ij}$ by Prop 9.2.14

Thus $\beta^{X,K}_m=\alpha_j$ by Lem 9.2.13 $\qquad i\,j\,\cdots$

and $\beta^{X,K}_1=\alpha_i$ by defn.

② For any $1\le n\le 2m$, let $i_n=i$ if $n=$odd,

and $i_n=j$ if n is even

Thus, by Prop 9.2.14, first part ⋯⋯

$id_X(s_is_j)^m(\alpha_i)\xlongequal{\text{by ①}}id_X s_{i_1}\cdots s_{i_m}s_{i_{m+1}}(\alpha_{i_{m+1}})$

$id_X\underbrace{(s_is_js_is_j\cdots s_is_j)}_{m\text{ 对}}(\alpha_i)$

$\underbrace{s_js_i\cdots s_j}_{m-1\text{个}}(\alpha_i)=\alpha_i$

$=-id_X s_{i_1}\cdots s_{i_m}(\alpha_{i_{m+1}})\xlongequal{\text{by ①, 9.2.14}}-id_X s_i(\alpha_i)=\alpha_i$

$id_X(s_is_j)^m(\alpha_j)=-id_X s_{i_1}\cdots s_{i_{2m+1}}(\alpha_j)$

$\xlongequal{\text{by ①, 9.2.14}}-id_X s_{i_1}\cdots s_{i_m}(\alpha_{i_m})=id_X s_{i_1}\cdots s_{i_{m-1}}(\alpha_{i_m})=\alpha_j$

This proves (1)

(2) by (1), $id_X(s_is_j)^m(\alpha_j)=\alpha_j$, fix α_i,α_j,

$(r_jr_i)^m(X)=X$ (CG4′) $\quad (F(id_X(s_is_j)^m)=id.)$

then $id_X(s_is_j)^m=id_X$

Prop 9.2.16 Assume that the semi-Cartan graph $\mathcal{G}$

satisfies (CG3′) and (CG4′)

Let $X\in\mathcal{X}$, $l\ge 1$, $K=(i_1,\cdots i_l)\in I^l$ and $i\in I$

s.t. K is X-reduced, and $id_X s_{i_1}\cdots s_{i_l}(\alpha_i)\notin\mathbb{N}_0^I$

Then there exists an X-reduced sequence

$(j_1,\cdots j_l)\in I^l$, s.t. $j_l=i$, and $id_X s_{i_1}\cdots s_{i_l}=id_X s_{j_1}\cdots s_{j_l}$

Rmk 9.2.17 (interesting!)

$ws_iw^{-1}=s_j$

(P38/Lem 3.10 [Kac 90] If α_i is a simple root

and $r_{i_1}\cdots r_{i_t}(\alpha_i)<0$,

then there exists s $(1\le s\le t)$ s.t.

$r_{i_1}\cdots r_{i_s}\cdots r_{i_t}r_i=r_{i_1}\cdots r_{i_{s-1}}r_{i_{s+1}}\cdots r_{i_t}$)

Proof of Prop 9.2.16:

① If $l=1$, then $i_1=i$, since $s_i^{r_i(X)}(\alpha_i)\notin\mathbb{N}_0^I$

generally, if $i_l=i$, then the Proposition

holds with $(j_1,\cdots j_l)=K$

② Assume $i_l\ne i$, Then $l\ge 2$

Let $\mathcal{M}$ be the set of pairs (K',p')

where $K'=(i'_1,\cdots i'_l)\in I^l$ is X-reduced

and $0\le p'<l$, s.t. $i'_l=i_l$, $i'_n\in\{i,i_l\}$, $\forall p'<n\le l$

$id_X s_{i_1}\cdots s_{i_l}=id_X s_{i'_1}\cdots s_{i'_l}$

Then $\mathcal{M}\ne\phi$, since $(K,l-1)\in\mathcal{M}$

Let $((k_1,\cdots k_l),p)\in\mathcal{M}$ with a smallest possible p

Then $\Lambda^X(k_1,\cdots,k_l)\subseteq\mathbb{N}_0^I$ by (CG3′)

In particular, $(k_1,\cdots k_p)$ is X-reduced by Lem 9.2.5

③ Let $j\in\{i,i_l\}$ and assume that

$id_X s_{k_1}\cdots s_{k_p}(\alpha_j)\notin\mathbb{N}_0^I$ —— 反证 goal

Then $p\ge 1$, by induction hypothesis

there exists $k'_1\cdots k'_p\in I$, s.t. $(k'_1\cdots k'_p)$ is -reduced

$k'_p=j$, and $id_X s_{k_1}\cdots s_{k_p}=id_X s_{k'_1}\cdots s_{k'_p}$

Let $K'=(k'_1\cdots k'_p, k_{p+1},\cdots k_l)$

Then $\Lambda^X(K')=\Lambda^X(k'_1\cdots k'_p)\cup\{\beta^{X,(k_1,\cdots k_l)}_n \mid p+1\le n\le l\}\subseteq\mathbb{N}_0^I$

and hence K' is X-reduced by Lem 9.2.5

Thus $(K',p-1)\in\mathcal{M}$, $\quad p$ min

contradiction!

④ $id_X s_{k_1}\cdots s_{k_p}(\alpha_j)\in\mathbb{N}_0^I$ $\quad\forall j\in\{i,i_l\}$

Then (9.2.4) $id_X s_{k_1}\cdots s_{k_p}(a\alpha_i+b\alpha_{i_l})\in\mathbb{N}_0^I$,

$\forall a,b\in\mathbb{N}_0$,

Let $Y=r_{i_p}\cdots r_{i_1}(X)$

Then $(k_{p+1}\cdots k_l)$ is Y-reduced and

$id_Y s_{k_{p+1}}\cdots s_{k_l}(\alpha_i)\in\mathbb{Z}\alpha_i+\mathbb{Z}\alpha_{i_l}\setminus\mathbb{N}_0^I$

$\mathbb{N}_0^I \ni id_X$ ⋯⋯

Thus $(k_{p+1}\cdots k_l, i)$ isn't Y-reduced by (CG3′)

and then $l-p=\bar{m}^Y_{k_{p+1},k_{p+2}}$

$id_Y s_{k_{p+1}}\cdots s_{k_l}=id_Y s_{k_{p+2}}\cdots s_{k_l}s_{k_{l+1}}$ by (CG4′), Lem 9.2.15

$(j_1,\cdots j_l)=(k_1\cdots k_p, k_{p+2}\cdots k_{l+1})$ $\qquad\square$